TRINITY COLLEGE

Semester Two Examination, 2020

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3&4

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

The time interval T between vehicles arriving at a 24-hour service station is known to follow an exponential distribution with a standard deviation of 42 seconds.

The mean of a random sample of 80 time intervals was 45 seconds.

(a) Use the sample to construct a 90% confidence interval for the mean of T. (4 marks)

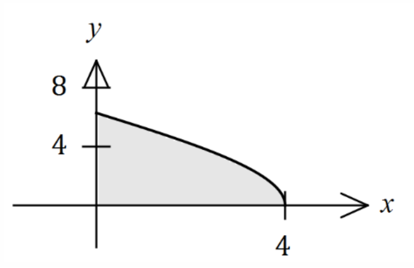
Solution
$z_{0.9} = 1.645$ $se = \frac{42}{\sqrt{80}} = 4.696$ $E = 1.645 \times 4.696 = 7.724$ $45 - 7.724 \leq \mu \leq 45 + 7.724$ $37.28 \leq \mu \leq 52.72$ s
Specific behaviours
✓ uses sample mean as interval centre ✓ calculates standard error ✓ uses correct z-score for confidence level ✓ correct bounds of interval

(b) State the key assumption made when constructing the interval in part (a) and explain how confident you are that the assumption is valid. (2 marks)

Solution
Assumed that the distribution of sample means $\bar{T}$ is normal. Although the population distribution is not normal, because the sample size is large (greater than 30) we can be confident that the distribution of sample means will approximate normality.
Specific behaviours
✓ states correct assumption ✓ states confident and refers to large sample size

Question 10 **(5 marks)**

The graph of $\frac{x}{4} = \cos\left(\frac{y}{4}\right)$ is shown for $0 \le y \le 2\pi$.



Show that when the shaded region bounded by the curve and the x-axis is rotated about the y-axis, the volume of revolution of the solid formed is $16\pi^2$.

Solution

$$V = \int \pi x^2 \, dy, \qquad x^2 = 16\cos^2\left(\frac{y}{4}\right)$$

$$\therefore V = \int_0^{2\pi} 16\pi \cos^2\left(\frac{y}{4}\right) dy, \qquad 2\cos^2\left(\frac{y}{4}\right) = 1 + \cos\left(\frac{y}{2}\right)$$

$$\begin{aligned} \therefore V &= 8\pi \int_0^{2\pi} 1 + \cos\left(\frac{y}{2}\right) dy \\ &= 8\pi \left[y + \sin\left(\frac{y}{2}\right)\right]_0^{2\pi} \\ &= 8\pi[2\pi + \sin\pi] - 8\pi[0 + \sin 0] \\ &= 8\pi[2\pi + 0] - 8\pi[0 + 0] \\ &= 16\pi^2 \end{aligned}$$

Specific behaviours

✓ integral for volume in terms of y
✓ uses double angle formula
✓ correctly antidifferentiates
✓ clearly shows all substitutions
✓ simplifies

Question 11 **(7 marks)**

The growth rate of electric vehicle (EV) sales as a percentage P of all passenger vehicle (PV) sales in Australia can be modelled by

$$\frac{dP}{dt} = rP(k - P)$$

At the start of 2013 ($t = 0$ years), EV sales were 0.03% of all PV sales in Australia. 4 years later, P had increased from 0.03% to $P = 0.21\%$. The maximum expected percentage of EV sales is 65%.

(a) Using a standard formula, or otherwise, show that $P \approx \dfrac{65}{1+2165.67e^{-0.4827t}}$. (3 marks)

Solution
Logistic equation with $k = 65$ and $P_0 = 0.03$:
$P = \dfrac{65(0.03)}{0.03 + (65 - 0.03)e^{-65rt}}$
Using $(4, 0.21)$:
$0.21 = \dfrac{1.95}{0.03 + 64.97e^{65\times 4r}} \Rightarrow r = 0.007495$
Hence
$P = \dfrac{1.95}{0.03 + 64.97e^{-0.4872t}}$
Specific behaviours
✓ uses logistic equation with k and P_0
✓ solves for r
✓ correct equation

Alternative Solution
Alt logistic equation, $k = 65$ and $P_0 = 0.03$:
$P = \dfrac{65}{1 + \left(\frac{65}{0.03} - 1\right)e^{-at}}$
Using $(4, 0.21)$:
$0.21 = \dfrac{65}{1 + 2165.\overline{6}e^{-4a}} \Rightarrow a = 0.4872$
Hence
$P = \dfrac{65}{1 + 2165.\overline{6}e^{-0.4872t}}$
Specific behaviours
✓ uses logistic equation with k and P_0
✓ solves for a
✓ correct equation

(b) Determine

(i) the percentage of EV sales expected at the start of 2025. (1 mark)

Solution
$P(12) = 8.55\%$
Specific behaviours
✓ correct percentage

(ii) the year in which EV sales are expected to reach 50% of all PV sales. (1 mark)

Solution
$P(t) = 50 \Rightarrow t = 18.2$. During 2031.
Specific behaviours
✓ correct year

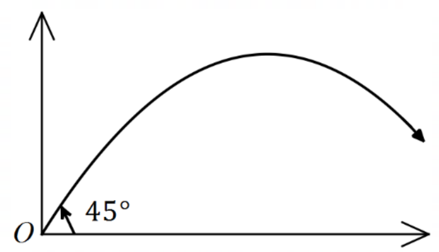
(c) State the year in which the growth rate of EV sales as a percentage of PV sales will reach a maximum and determine this maximum growth rate. (2 marks)

Solution
Require $P = 65 \div 2 = 32.5 \Rightarrow t = 15.77$. During 2028.
$\dot{P} = 0.007495(32.5)^2 = 7.92\%$ per year
Specific behaviours
✓ correct year
✓ correct growth rate

Question 12 **(6 marks)**

A small projectile is launched upwards from O at an angle of $45°$ to the horizontal, with an initial speed of 24 m/s.

The motion of the projectile is only affected by gravity, so that the acceleration at any time t seconds is given by $\mathbf{a}(t) = -9.8\mathbf{j}$ m/s^2.



(a) Show that the position vector of the projectile relative to O after t seconds is given by $\mathbf{r}(t) = \left(12\sqrt{2}t\right)\mathbf{i} + \left(12\sqrt{2}t - 4.9t^2\right)\mathbf{j}$ m. (3 marks)

Solution

$$\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j}$$

$$\mathbf{v}(0) = (24\cos 45°)\mathbf{i} + (24\sin 45°)\mathbf{j} = \left(12\sqrt{2}\right)\mathbf{i} + \left(12\sqrt{2}\right)\mathbf{j}$$

$$\mathbf{v}(t) = \int -9.8\mathbf{j}\,dt$$
$$= \left(12\sqrt{2}\right)\mathbf{i} + \left(12\sqrt{2} - 9.8t\right)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt, \qquad \mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j}$$
$$= \left(12\sqrt{2}t\right)\mathbf{i} + \left(12\sqrt{2}t - 4.9t^2\right)\mathbf{j}$$

Specific behaviours

✓ antidifferentiates acceleration
✓ uses $v(0)$ for both constants
✓ antidifferentiates velocity

(b) Determine the maximum altitude of the projectile above O and the time taken to reach this altitude. (3 marks)

Solution

Maximum altitude when

$$12\sqrt{2} - 9.8t = 0$$
$$t = \frac{60\sqrt{2}}{49} \approx 1.732 \text{ s}$$

$$h = 12\sqrt{2}t - 4.9t^2\Big|_{t=1.732}$$
$$= \frac{720}{49} \approx 14.7 \text{ m}$$

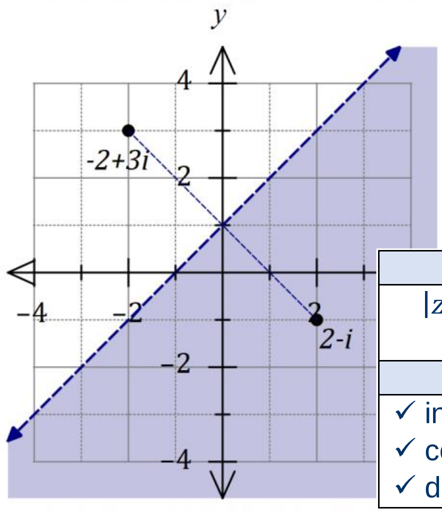
Specific behaviours

✓ time to reach maximum
✓ uses $\mathbf{j}$ coefficient of position
✓ maximum altitude

Question 13 **(9 marks)**

(a) Sketch the locus of the complex number $z = x + iy$ given by

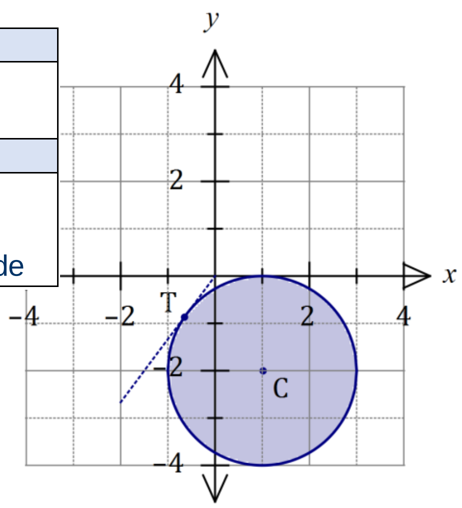
(i) $|z + 2 - 3i| > |z - 2 + i|$. (3 marks)



Solution
$\lvert z - (-2 + 3i)\rvert > \lvert z - (2 - i)\rvert$ See diagram
Specific behaviours
✓ indicates two points ✓ correct location of bisector ✓ dotted line and correct shading

(ii) $|z - 1 + 2i| \le 2$. (3 marks)

Solution
$\lvert z - (1 - 2i)\rvert \le 2$ See diagram
Specific behaviours
✓ indicates centre ✓ circle, radius 2 ✓ solid line and shades inside



(b) For the locus $|z - 1 + 2i| \le 2$ determine, correct to the nearest degree, the minimum value of $\arg(z)$, $-180° < \arg(z) \le 180°$. (3 marks)

Solution
Minimum argument is angle tangent OT (shown on diagram) makes with positive x-axis. $CT = 2, CO = \sqrt{5} \Rightarrow \angle COT = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \approx 63.4°$ Using symmetry, minimum $\arg(z) = -2 \times \angle COT = -127°$
Specific behaviours
✓ indicates ray on diagram ✓ calculates angle in triangle ✓ correct argument

Question 14 **(10 marks)**

The shell weight W of eggs laid by hens in a flock is known to be normally distributed with mean of 6.54 g and standard deviation 0.56 g.

(a) A random sample of 40 eggs is selected from the flock and the mean shell weight of these eggs calculated.

(i) State the distribution of $\bar{W}$, the sample mean. (3 marks)

Solution
Sample size is 40, which is > 30, so sample means will be approximately normally distributed $$\bar{W} \sim N\left(6.54, \frac{0.56^2}{40}\right) \sim N(6.54, 0.00784)$$ Mean of $\bar{W}$ is 6.54 g and variance is 0.00784 g^2 (sd ≈ 0.0885 g).
Specific behaviours
✓ states sample mean is normally distributed ✓ states correct mean of distribution ✓ states correct variance (or sd) of distribution

(ii) Determine the probability that the sample mean is between 6.5 g and 6.6 g. (2 marks)

Solution
$$P(6.5 < \bar{W} < 6.6) = P(-0.4518 < z < 0.6776) = 0.4253$$
Specific behaviours
✓ indicates correct z-scores ✓ correct probability

(iii) Suppose the size of the random sample was halved. Explain, without any further calculation, how this will affect your answer to part (a)(ii). (2 marks)

Solution
The probability will decrease. With half the sample size, the variance of the sampling distribution will double and so the range of z-scores for the interval will be smaller, leading to a lower probability.
Specific behaviours
✓ states probability will decrease ✓ justifies with increased spread or smaller range of z-scores

(b) Random samples of n eggs were repeatedly selected from the flock and the mean weight of each sample recorded. It was observed that 8% of the sample means weighed less than 6.44 g. Determine the value of n. (3 marks)

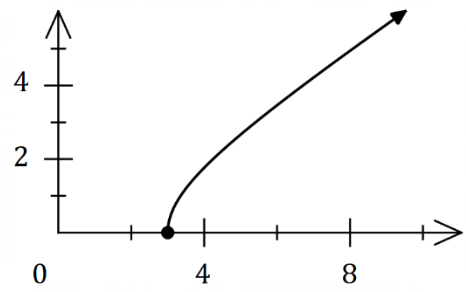
Solution
$P(\overline{W} < 6.5) = P(z < k) = 0.08 \Rightarrow k = -1.4051$ $\dfrac{6.44 - 6.54}{s} = -1.4051 \Rightarrow s = 0.07117$ $\dfrac{0.56}{\sqrt{n}} = 0.07117 \Rightarrow n = 62$
Specific behaviours
✓ indicates z-score for required probability   ✓ indicates calculation for sample sd   ✓ correct value of n as integer

Question 15 **(8 marks)**

The path of a particle is shown in the diagram.

Its position, in metres, relative to the origin O at time t seconds is given by

$\mathbf{r}(t) = 3\sec(t)\,\mathbf{i} + 2\tan(t)\,\mathbf{j},\ 0 \le t < \frac{\pi}{2}.$



(a) Determine the Cartesian equation of the path of the particle. (3 marks)

Solution
$x = 3\sec(t)\,, \quad y = 2\tan(t)$ $\frac{x^2}{9} = \sec^2 t\,, \quad \frac{y^2}{4} = \tan^2 t$ $\therefore \frac{x^2}{9} = \frac{y^2}{4} + 1$, where $x \ge 3$ and $y \ge 0$.
Specific behaviours
✓ indicates use of identity   ✓ correct Cartesian equation (any transposition)   ✓ restricts domain

(b) Determine the exact speed of the particle when $t = \frac{\pi}{3}$. (3 marks)

Solution
$\mathbf{v}(t) = (3\sin(t)\sec^2(t))\mathbf{i} + 2(\sec^2(t))\mathbf{j}$ $\mathbf{v}\left(\frac{\pi}{3}\right) = 6\sqrt{3}\mathbf{i} + 8\mathbf{j}$ $\therefore$ speed $= 2\sqrt{43}$ m/s
Specific behaviours
✓ derivative of position   ✓ velocity at required time   ✓ exact speed

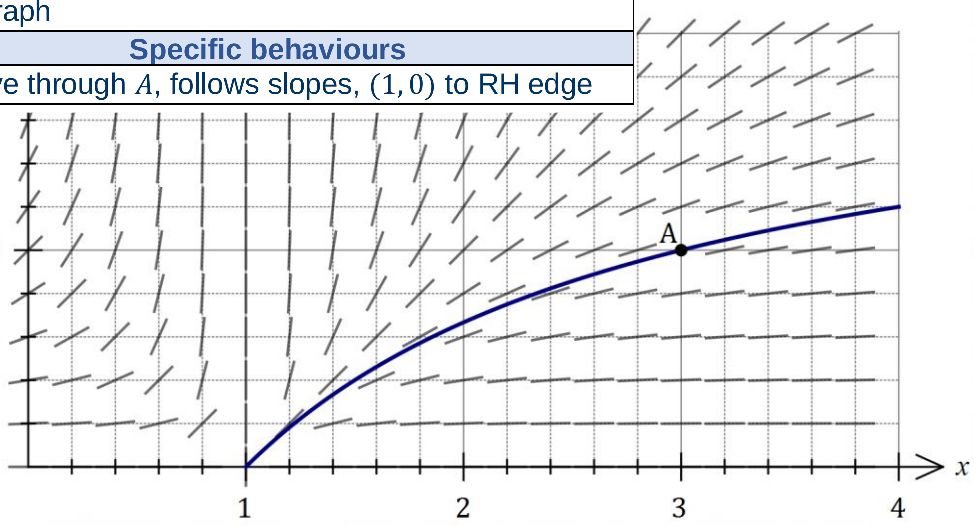
(c) Determine, correct to two decimal places, the distance the particle travels between $t = 0$ and $t = \frac{\pi}{3}$. (2 marks)

Solution
$\int_0^{\frac{\pi}{3}} \lvert\mathbf{v}(t)\rvert\, dt = 4.71$ m
Specific behaviours
✓ indicates method   ✓ correct distance

Question 16 **(6 marks)**

The graph below shows the slope field for the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{(x-1)^2}$.

Solution
See graph
Specific behaviours
✓ curve through A, follows slopes, $(1,0)$ to RH edge



(a) Determine the value of the slope field at point A. (1 mark)

Solution
$y' = \dfrac{1^2}{(3-1)^2} = \dfrac{1}{4}$
Specific behaviours
✓ correct value

(b) On the axes above, sketch the solution curve for the differential equation that passes through point A. (1 mark)

(c) Determine the particular solution $y = f(x)$ to the differential equation that has initial solution $f(3) = 1$. (4 marks)

Solution
$\displaystyle\int \frac{1}{y^2}\,dy = \int \frac{1}{(x-1)^2}\,dx$ $\dfrac{1}{y} = \dfrac{1}{x-1} + c$ $(3,1) \Rightarrow 1 = \dfrac{1}{2} + c \Rightarrow c = \dfrac{1}{2}$ $\dfrac{1}{y} = \dfrac{1}{x-1} + \dfrac{1}{2} = \dfrac{2+(x-1)}{2(x-1)}$ $y = \dfrac{2x-2}{x+1}$
Specific behaviours
✓ separates variables   ✓ antidifferentiates both sides with constant   ✓ evaluates constant   ✓ expresses in form $y = f(x)$

Question 17 **(10 marks)**

Lines L_1 and L_2 have equations $\mathbf{r} = \begin{pmatrix} 14 \\ -6 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 6 \\ -7 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$ respectively, they both lie in plane Π and they intersect at point P.

(a) Determine coordinates of point P. (3 marks)

Solution
Equating $\mathbf{i}$ and $\mathbf{j}$ coefficients: $$14 + 4\lambda = 6 - 2\mu, \qquad -6 - \lambda = -7 + 2\mu$$ Solving simultaneously: $$\lambda = -3, \mu = 2$$ No need to check for $\mathbf{k}$ coefficients as told lines intersect. Hence $\overrightarrow{OP} = \begin{pmatrix} 6 \\ -7 \\ -5 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ or $P(2, -3, 1)$.
Specific behaviours
✓ equates coefficients ✓ solves simultaneously ✓ coordinates of P

(b) Determine the Cartesian equation of plane Π. (4 marks)

Solution
Normal to plane: $$\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$$ Hence $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $k = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 10$ Equation of plane: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 10.$ $$x - 2y + 2z = 10$$
Specific behaviours
✓ uses directions of lines in cross product ✓ correct cross product ✓ indicates use of dot product to obtain constant ✓ equation in Cartesian form

Sphere S has a radius of 12, is tangential to plane Π at point P and the origin lies within it.

(c) Determine the vector equation of sphere S. (3 marks)

Solution
Centre of sphere: $$\overrightarrow{OC} = \overrightarrow{OP} \pm 12\hat{\mathbf{n}}$$ $$\overrightarrow{OC_1} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \frac{12}{3} \times \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ 9 \end{pmatrix}$$ $$\overrightarrow{OC_2} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} - \frac{12}{3} \times \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -7 \end{pmatrix}$$ For sphere to contain O require $\lvert\overrightarrow{OC}\rvert < 12$: $$\left\lvert\overrightarrow{OC_2}\right\rvert = \sqrt{78} \approx 8.83$$ Equation of S: $$\left\lvert \mathbf{r} - \begin{pmatrix} -2 \\ 5 \\ -7 \end{pmatrix} \right\rvert = 12$$
Specific behaviours
✓ indicates method to locate both centres ✓ indicates method to check contains origin ✓ correct equation *(NB Max 1/3 if no check for origin seen)*

Question 18 **(8 marks)**

Let $u = -2 + \sqrt{12}i$ and $v = 1 + i$.

(a) Solve the equation $z^3 = 2u$, giving all solutions in the form $r \operatorname{cis} \theta, -\pi < \theta \le \pi$. (3 marks)

Solution
$z^3 = 4(-1+\sqrt{3}i) = 8 \operatorname{cis}\left(\frac{2\pi}{3}\right)$ $z = 2\operatorname{cis}\left(\frac{2\pi}{9} \pm \frac{2n\pi}{3}\right)$ $z = 2\operatorname{cis}\left(\frac{2\pi}{9}\right), z = 2\operatorname{cis}\left(\frac{8\pi}{9}\right), z = 2\operatorname{cis}\left(-\frac{4\pi}{9}\right)$
Specific behaviours
✓ expresses $2u$ in polar form   ✓ one correct solution   ✓ all correct solutions

(b) Express w in both Cartesian and polar forms, where $w = u \div v$. (2 marks)

Solution
$w = \sqrt{3} - 1 + (\sqrt{3}+1)i$ $= 2\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$
Specific behaviours
✓ correct Cartesian form   ✓ correct polar form

(c) Show how to use your answers from part (b) to determine an exact value for

(i) $\sin\left(\frac{5\pi}{12}\right)$. (2 marks)

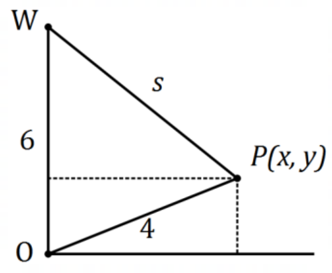
Solution
$w = 2\sqrt{2}\cos\left(\frac{5\pi}{12}\right) + 2\sqrt{2}\sin\left(\frac{5\pi}{12}\right)i = \sqrt{3} - 1 + (\sqrt{3}+1)i$ Equating imaginary parts: $2\sqrt{2}\sin\left(\frac{5\pi}{12}\right) = \sqrt{3}+1$ $\sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$
Specific behaviours
✓ equates imaginary parts of w in both forms   ✓ shows division by modulus to obtain value

(ii) $\tan\left(\frac{5\pi}{12}\right)$. (1 mark)

Solution
$\tan\left(\frac{5\pi}{12}\right) = \frac{\operatorname{Im} w}{\operatorname{Re} w} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$
Specific behaviours
✓ indicates quotient of Im and Re parts to obtain value

Question 19 **(7 marks)**

A light rope from winch W, at the top of a six-metre-tall wall OW, is attached to point P at the end of a four-metre-long pole OP.



The winch is winding the rope in at a rate of 15 cm/s so that the pole is rotating about O from a horizontal to a vertical position.

Let s be the length of the rope PW, and let x be the horizontal distance and y be the vertical distance of P relative to O.

Show that $s^2 = 52 - 12y$ and hence determine the rate at which x is decreasing at the instant that y is increasing at a rate of 10 cm/s.

Solution

Using right triangle with s as hypotenuse:

$$\begin{aligned} s^2 &= x^2 + (6-y)^2 \\ &= 16 - y^2 + 36 - 12y + y^2 \\ &= 52 - 12y \end{aligned}$$

Given: $\frac{ds}{dt} = -0.15$ and $\frac{dy}{dt} = 0.1$. Required to find $\frac{dx}{dt}$.

Using length of pole:

$$x^2 + y^2 = 4^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

Using $s^2 = 52 - 16y$:

$$2s\frac{ds}{dt} = -12\frac{dy}{dt} \Rightarrow s = \frac{-6 \times 0.1}{-0.15} = 4$$

When $s = 4, y = (52 - 4^2) \div 12 = 3$ and $x = \sqrt{4^2 - 3^2} = \sqrt{7}$

$$\frac{dx}{dt} = -\frac{3}{\sqrt{7}}(10) = -\frac{30\sqrt{7}}{7} \approx -11.3$$

Hence x is decreasing at a rate of $\frac{30\sqrt{7}}{7}$ cm/s.

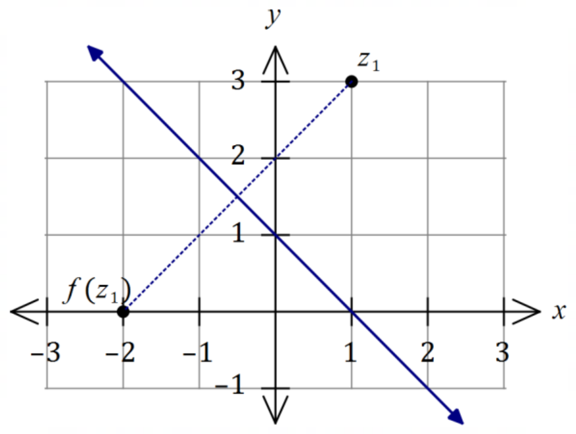
Specific behaviours

✓ derives expression for s^2 in terms of y
✓ uses length of pole to relate x and y
✓ implicitly differentiates to relate $\frac{dx}{dt}$ and $\frac{dy}{dt}$
✓ implicitly differentiates to relate $\frac{ds}{dt}$ and $\frac{dy}{dt}$
✓ solves for s
✓ solves for x and y
✓ states required rate with units

Question 20 **(8 marks)**

Consider the function $f(z) = -i\bar{z} + 1 + i$, where $z = x + iy$ and $x, y \in \mathbb{R}$.

(a) Determine $f(z_1)$ when $z_1 = 1 + 3i$ and use the Argand diagram below to show that $f(z_1)$ is a reflection of z_1 in the line $x + y = 1$. (3 marks)



Solution
$f(1+3i) = -2$ See diagram
Specific behaviours
✓ evaluates function ✓ plots z_1 and $f(z_1)$ ✓ line, indicates reflection

Any reflection of z in the complex plane can be expressed in the form $g(z) = a\bar{z} + b$, where a and b are complex constants.

(b) By considering the transformations of the axes intercepts of the reflection line, or otherwise, determine the value of a and the value of b so that $g(z)$ represents a reflection of z in the line $y = 3x - 3$. (4 marks)

Solution
Intercepts: $x = 0, y = -3$ and $y = 0, x = 1$. $g(1) = a + b$ and $g(-3i) = 3ai + b$. If a point lies on mirror line, then $g(z) = z$. $a + b = 1$ and $3ai + b = -3i$ Solving simultaneously gives $a = \dfrac{-4+3i}{5} = -0.8 + 0.6i$ $b = \dfrac{9-3i}{5} = 1.8 - 0.6i$
Specific behaviours
✓ intercepts ✓ forms two equations in a and b ✓ solves one constant ✓ solves both constants

(c) Given $w = 20 - 18i$, determine w', the reflection of w in the line $y = 3x - 3$. (1 mark)

Solution
$w' = g(20 - 18i) = -25 - 3i$
Specific behaviours
✓ correct complex number

Question 21 **(8 marks)**

Particles P and Q travel in a straight line with displacement x m and velocity v m/s at time t s.

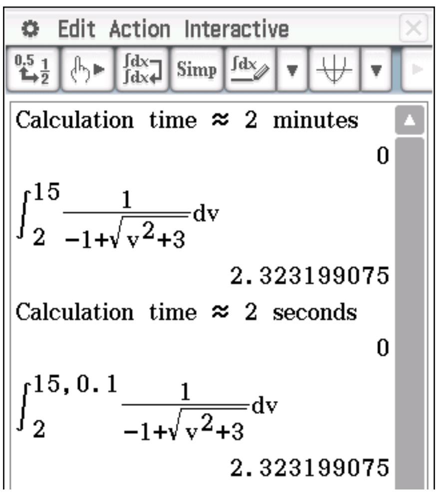
(a) The acceleration of P is given by $a = -\frac{v}{5}$ m/s^2, and when $t = 0$, $x = 0$ and $v = 24$. Determine, correct to one decimal place, the displacement of P after 4 seconds.

(5 marks)

Solution
$a = \frac{dv}{dt} = -\frac{1}{5}v$ $\int \frac{1}{v} dv = \int -\frac{1}{5} dt$ $\ln\lvert v\rvert = -0.2t + c$ $v = ke^{-0.2t}$ $(0, 24) \Rightarrow v = 24e^{-0.2t}$ $v = \frac{dx}{dt} = 24e^{-0.2t}$ $x = \int_0^4 24e^{-0.2t}\, dt$ $= 66.1$ m
Specific behaviours
✓ uses $\frac{dv}{dt}$ and separates variables   ✓ expression for v with constant   ✓ correct expression for v ✓ uses $\frac{dx}{dt}$ to obtain integral for x ✓ correct displacement

(b) The acceleration of Q is given by $a = -1 + \sqrt{v^2 + 3}$. Determine, correct to two decimal places, the time taken for its velocity to increase from 2 m/s to 15 m/s. (3 marks)

Solution
$a = \frac{dv}{dt} = -1 + \sqrt{v^2 + 3}$ $\therefore \frac{dt}{dv} = \frac{1}{-1 + \sqrt{v^2 + 3}}$ Net change Δt is integral of rate of change: $\Delta t = \int_2^{15} \frac{1}{-1 + \sqrt{v^2 + 3}} dv$ $= 2.32$ s
Specific behaviours
✓ expression for $\frac{dt}{dv}$ ✓ integral for net change   ✓ correct time



End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________